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Best Uniform Approximation by Linear Fractional Transformations

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1. INTRODUCTION

Let $f(x)$ be a real-valued function defined on the closed interval $[-1, 1]$. We shall assume throughout this paper that f is an even function and that

$$0 = f(0) \leqslant f(x) \leqslant f(1) = 1, \qquad -1 \leqslant x \leqslant 1.$$

We consider the problem of determining the best uniform approximation to f on $[-1, 1]$ by linear fractional transformations

$$U(x) = \frac{ax + b}{cx + d}.$$

Here x is a real variable and a, b, c and d are complex numbers (we exclude once and for all the case in which both c and d are zero). In general such a transformation U takes values in the extended complex plane. If U takes only (extended) real values, we shall say that U is a *real* transformation; if $U(x) = \overline{U(-x)}$ for all x then U will be called *symmetric* (as usual, $\bar{z}$ denotes the complex conjugate of z).

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For each given function f our problem is to minimize the quantity

$$\| U - f\|_\infty = \sup_{-1 \leqslant x \leqslant 1} | U(x) - f(x)|$$

over various classes of linear fractional transformations U. Specifically, we want to determine the degrees of approximation

$$E_R(f) = \inf\{\| U - f\|_\infty : U \text{ is real}\},$$
$$E_S(f) = \inf\{\| U - f\|_\infty : U \text{ is symmetric}\},$$
$$E_C(f) = \inf\{\| U - f\|_\infty : U \text{ is arbitrary}\},$$

and to identify the extremal transformations whenever they exist.

In the next section we obtain the inequalities

$$1/4 \leqslant E_C(f) \leqslant E_S(f) \leqslant E_R(f) = 1/2. \tag{1.1}$$

Thus $E_R(f)$ is completely determined for all f. However, little seems to have been previously known about $E_C(f)$ or $E_S(f)$ even when f is well behaved. In fact, the present research arose from questions posed by R. S. Varga (Conference on Approximation Theory, University of California, Riverside, February, 1976) stemming from his work with E. B. Saff ([2], [3], [4]). One aspect of their work regards approximating real-valued functions on real intervals by complex rational functions (cf. [3]). For example, they were interested in determining $E_C(f_2)$, were $f_2(x) = x^2$. Saff and Varga have determined by example in [3] that

$$E_C(f_2) \leqslant \sqrt{2} - 1 = .4142 \cdots,$$

so that one does indeed do better than $1/2$ by considering complex linear fractional transformations.

From more general results developed below we shall show that the degree of symmetric approximation is given by

$$E_S(f_2) = (4/27)^{1/2} = 0.38490018 \cdots,$$

and that this degree of approximation is attained by exactly two symmetric conjugate transformations. This provides a nontrivial upper bound for $E_C(f_2)$, but we suspect that much more is true. Namely, that any function f can be approximated as well by symmetric transformations as by arbitrary ones. Thus we make the following conjecture.

CONJECTURE. For any function f, $E_C(f) = E_S(f)$.

In the present paper we shall consider mainly approximation by symmetric transformations. Once the preliminary results of Section 2 have been established, the remainder of the paper is divided into three parts.

In Section 3 we obtain a lower bound for $E_S(f)$ by first approximating $f(x)$ on the finite set $Z(\omega) = \{-1, -\omega, \omega, 1\}$, where $0 \leqslant \omega \leqslant 1$. Corresponding to each ω in the open interval $(0, 1)$ we define a particular symmetric transformation, denoted by $U(x; f, \omega)$, which is a good approximation to f on $Z(\omega)$. Specifically, for each $x \in Z(\omega)$, $0 < \omega < 1$, we have

$$| U(x; f, \omega) - f(x)| = \delta(f, \omega), \tag{1.2}$$

where

$$\delta(f, \omega) = \frac{\omega^{1/2}(1 - f(\omega))}{1 + \omega}. \tag{1.3}$$

Our first result (Theorem A) shows that, modulo complex conjugation, no other symmetric transformation produces as good an approximation to f on $Z(\omega)$.

THEOREM A. *Let f be given, let $\omega \in [0, 1]$ be fixed and suppose that U is any symmetric transformation. Then*

$$\max_{x \in Z(\omega)} | U(x) - f(x)| \geqslant \delta(f, \omega). \tag{1.4}$$

If $\omega \in (0, 1)$ then equality holds in (1.4) *if and only if $U(x) = U(x; f, \omega)$ or $\overline{U(x)} = U(x; f, \omega)$.*

Next we define

$$\Delta(f) = \sup_{0 \leqslant \omega \leqslant 1} \delta(f, \omega). \tag{1.5}$$

From Theorem A we obtain the following lower bound.

THEOREM B. *Let f be given and let U be any symmetric transformation. Then*

$$\| U - f\|_\infty \geqslant \max(1/4, \Delta(f)) \tag{1.6}$$

and hence

$$E_S(f) \geqslant \max(1/4, \Delta(f)). \tag{1.7}$$

In Section 4 we determine a class of functions f which satisfy

$$E_S(f) = \Delta(f) = \| U - f\|_\infty \tag{1.8}$$

for some symmetric transformation U. In order to achieve this we shall need to impose some restrictions on f.

CONDITION 1. *The function f is continuous on* $[-1, 1]$, *is differentiable on* $(0, 1)$ *and* $f'(x) \geqslant 0$ *for each* $x \in (0, 1)$.

One consequence of the continuity of f is that (1.8) can hold only if the supremum in (1.5) is attained at a unique point $\omega = \Omega$ in $(0, 1)$. Hence, we can restrict our attention to those functions f for which the following condition holds.

CONDITION 2. *There exists a unique point* $\Omega \in (0, 1)$, *depending only on* f, *such that* $\Delta(f) = \delta(f, \Omega)$.

We shall show that Condition 2 is satisfied if, for instance, f is continuous and convex (cf. Theorem 4.2), or if $f(x) = |x|^\alpha$ for any $\alpha > 0$ (cf. Theorem 5.1).

We now state the main result of Section 4.

THEOREM C. *Let f be a function satisfying Conditions* 1 *and* 2 *and let U be any symmetric transformation. Suppose in addition that the function*

$$x \to \frac{(x^2 + \Omega)^2 f'(x)}{x} \tag{1.9}$$

is increasing on $(0, 1)$. *Then* $1/4 < \Delta(f)$ *and*

$$\| U - f \|_\infty \geqslant \Delta(f) \tag{1.10}$$

with equality in (1.10) *if and only if* $U(x) = U(x; f, \Omega)$ *or* $\overline{U(x)} = U(x; f, \Omega)$. *In particular we have* $E_S(f) = \Delta(f)$.

In the final section of the paper we consider approximation of the special class of functions $f_\alpha(x) = |x|^\alpha$, where $\alpha > 0$. It is easily verified that Conditions 1 and 2 hold for all $\alpha > 0$ and we denote the corresponding value of Ω by Ω_α. If $\alpha \geqslant 2$ then f_α also satisfies the additional hypothesis in Theorem C and thus its degree of symmetric approximation and the precise extremal transformations are completely determined. The case $\alpha = 2$ gives our partial answer to Saff and Varga's original questions concerning $E_C(f_2)$.

If $0 < \alpha < 2$ then the hypothesis in Theorem C fails to hold. Nevertheless the conclusion of Theorem C continues to hold for certain values of α. Let $\kappa = 1.4397589 \cdots$ be the unique solution in the interval $(1, \infty)$ of the equation

$$(2\kappa - 1)^{2\kappa-1} = \frac{\kappa}{\kappa - 1}. \tag{1.11}$$

We prove the following result in Section 5.

THEOREM D. *Let* $f_\alpha(x) = |x|^\alpha$, *where* $\alpha > 0$ *and let* U *be any symmetric transformation.*

(i) *If* $\kappa \leqslant \alpha$ *then* $1/4 < \Delta(f_\alpha)$ *and*

$$\| U - f_\alpha \|_\infty \geqslant \Delta(f_\alpha)$$

with equality if and only if $U(x) = U(x; f_\alpha, \Omega_\alpha)$ *or* $\overline{U(x)} = U(x; f_\alpha, \Omega_\alpha)$. *In particular* $E_S(f_\alpha) = \Delta(f_\alpha)$.

(ii) *If* $0 < \alpha < \kappa$ *then*

$$\| U - f_\alpha \|_\infty > \max(1/4, \Delta(f_\alpha)). \tag{1.12}$$

At the end of Section 5 we give (Table 1) some numerical values for Ω_α and $\Delta(f_\alpha)$. Theorem 5.4 states that the constant κ is transcendental.

We have been informed by A. Ruttan [3] that he has proved the conjecture $E_C(f) = E_S(f)$ in certain cases. In generalizing our results he has shown that the inequalities (1.4), (1.6), (1.10) and (1.12) of Theorems A, B, C, and D, respectively, are valid for any complex linear fractional transformation U. Thus, in the instances where we have explicitly determnied $E_S(f)$, it is true that $E_C(f) = E_S(f)$. However, in the class of general complex linear fractional transformations the exact extremal transformations are still unknown. For instance, under the hypotheses of Theorems C or D it is yet undetermined whether or not there are *non*-symmetric transformations with the same degree of approximations as $U(x; f, \Omega)$ and $\overline{U(x; f, \Omega)}$. Indeed, Ruttan proves the existence of an even continuous real valued function f on $[-1, 1]$ with a continuum of best approximations from the class of linear fractional transformations.

In view of the results presented here it is natural to ask whether similar phenomena occur when approximation by rational functions of higher orders is allowed. In this connection, E. B. Saff and R. S. Varga [3] have recently constructed examples where the approximation by complex rational functions (of class $R_{n,n}$, $n = 1, 2, \ldots$) is, once again, better than that attainable in the real situation. The determination of the degree of approximation in such cases remains an interesting open problem.

Our initial investigations (at the California Institute of Technology) made use of a computer program developed by H. F. Bohnenblust. We wish to thank Professors Bohnenblust, R. A. Dean and K. W. Holladay for their valuable assistance in this regard.

2. Preliminaries

We begin with a technical lemma in order to establish the estimates in (1.1). The lemma has a simple geometric proof which we omit.

LEMMA 2.1. *Let f be given and suppose $U(x) = (ax + b)/(cx + d)$ satisfies one of the following conditions*:

(i) $ad - bc = 0$,

(ii) $d = 0$,

(iii) $c = 0$,

(iv) d/c *is real and* $|d/c| > 1$.

Then for any $\omega \in [0, 1]$

$$\max_{x \in Z(\omega)} |U(x) - f(x)| \geqslant \tfrac{1}{2}(1 - f(\omega)), \tag{2.1}$$

with equality if and only if $U(x) \equiv \frac{1}{2}(1 + f(\omega))$.

If U is a real transformation then it maps $[-1, 1]$ into the real line. This observation leads to the following theorem.

THEOREM 2.2. *Let f be given and let U be any real transformation. Then* $\| U - f \|_\infty \geqslant 1/2$ *with equality if and only if* $U(x) \equiv 1/2$.

THEOREM 2.3. *Let f be given and let U be any transformation. Then*

$$\| U - f \|_\infty > 1/4. \tag{2.2}$$

Proof. Let $U(x) = (ax + b)/(cx + d)$ and suppose that

$$\| U - f \|_\infty \leqslant 1/4. \tag{2.3}$$

Letting $\omega = 0$ in (2.1) we see that none of the conditions (i)–(iv) of Lemma 2.1 can hold. Also, if d/c is real and $|d/c| \leqslant 1$, then $\| U - f \|_\infty = \infty$. Hence, $ad - bc \neq 0$, $c \neq 0$, $d \neq 0$ and d/c is not real, implying U maps $[-1, 1]$ bijectively onto the arc of a circle. This geometric fact evidently contradicts (2.3) and the proof is complete.

When f is continuous the degree of approximation $E_C(f)$ is always attained [6, p. 351]. Hence, $1/4 < E_C(f)$. However, if continuity is dropped then $E_C(f)$ may equal $1/4$. (For an example, let $f(x) \equiv 1/2$ except for $f(0) = 0$, $f(\pm 1) = 1$. Consider the transformations $U_\eta(x) = (3x - i\eta)/(4x - 4i\eta)$ for $0 < \eta \leqslant 1/2$.)

For the symmetric transformations that map $[-1, 1]$ onto an arc of a circle there is a convenient change of parameters.

THEOREM 2.4. *$U(x) = (ax + b)/(cx + d)$ is a symmetric transformation satisfying $ad - bc \neq 0$, $c \neq 0$ and $d \neq 0$ if and only if U has the form*

$$U(x) = s + r\left(\frac{x + it}{x - it}\right), \tag{2.4}$$

where r, s and t are uniquely determined real numbers with r and t nonzero.

Proof. Straightforward.

3. A Lower Bound for $E_S(f)$

For each given function f and each $\omega \in (0, 1)$ we denote by $U(x; f, \omega)$ the symmetric transformation defined as follows. If $f(\omega) = 1$ then $U(x; f, \omega)$ is the constant transformation 1. If $0 \leqslant f(\omega) < 1$, then

$$U(x; f, \omega) = s + r\,\frac{x + it}{x - it}, \tag{3.1}$$

where the parameters r, s and t depend on f and ω according to

$$\begin{aligned} r &= r(f, \omega) = \frac{(1 - \omega)(1 - f(\omega))}{2(1 + \omega)}, \\ s &= s(f, \omega) = \frac{1 + f(\omega)}{2}, \\ t &= t(f, \omega) = \omega^{1/2}. \end{aligned} \tag{3.2}$$

We remark that if $0 < \omega < 1$, then (1.2) can be verified for the transformation $U(x; f, \omega)$ by a trivial calculation.

Proof of Theorem A. We may assume that $0 < \omega < 1$ and $0 < f(\omega) < 1$ as (1.4) is obvious in the other cases. From (1.3) we observe that $\delta(f, \omega) < \frac{1}{2}(1 - f(\omega))$. Hence, if U is any symmetric transformation satisfying

$$\max_{x \in Z(\omega)} |U(x) - f(x)| \leqslant \delta(f, \omega), \tag{3.3}$$

then Lemma 2.1 and Theorem 2.4 show that

$$U(x) = s_0 + r_0\left(\frac{x + it_0}{x - it_0}\right), \tag{3.4}$$

where r_0, s_0, and t_0 are real with r_0 and t_0 nonzero. The theorem will therefore be proved if we show that $r_0 = r$, $s_0 = s$, and $(t_0)^2 = t^2$ where r, s and t are given by (3.2). This shows that $U(x) = U(x; f, \omega)$ or $\overline{U}(x) = U(x; f, \omega)$.

We introduce the perturbations ζ, η and ξ given by

$$s_0 = s + \zeta, \qquad r_0 = r + \eta, \qquad (t_0)^2 = t^2 + \xi. \tag{3.5}$$

Then ζ, η and ξ are real with

$$t^2 + \xi = \omega + \xi > 0. \tag{3.6}$$

From (3.3) we have

$$| U(\omega) - f(\omega)|^2 \leqslant \delta^2. \tag{3.7}$$

Expanding (3.7) and using (3.4) we obtain the inequality

$$\omega^2\{f(\omega) - r_0 - s_0\}^2 + (t_0)^2\{f(\omega) + r_0 - s_0\}^2 \leqslant \delta^2\{\omega^2 + (t_0)^2\}.$$

Substituting the values from (3.2) and (3.5) we have

$$\omega^2 \left\{f(\omega) - \frac{1 + \omega f(\omega)}{1 + \omega} - \zeta - \eta\right\}^2 + (\omega + \xi) \left\{f(\omega) - \frac{\omega + f(\omega)}{1 + \omega} - \zeta + \eta\right\}^2$$
$$\leqslant \delta^2\omega^2 + \delta^2(\omega + \xi).$$

Then using (1.3) we obtain

$$\omega^2\{\omega^{-1/2}\delta + \zeta + \eta\}^2 + (\omega + \xi)\{\omega^{1/2}\delta + \zeta - \eta\}^2$$
$$\leqslant \delta^2\omega^2 + \delta^2(\omega + \xi). \tag{3.8}$$

Similarly, starting from

$$| U(1) - 1 |^2 \leqslant \delta^2,$$

we deduce the inequality

$$\{\omega^{1/2}\delta - \zeta - \eta\}^2 + (\omega + \xi)\{\omega^{-1/2}\delta - \zeta - \eta\}^2$$
$$\leqslant \delta^2 + \delta^2(\omega + \xi). \tag{3.9}$$

When (3.9) is multiplied through by ω and then added to (3.8), the resulting inequality is

$$(2\omega + \xi)\, \zeta^2 - 2\xi\zeta\eta + (2\omega + \xi)\, \eta^2 \leqslant 0. \tag{3.10}$$

The left-hand side of (3.10) is a quadratic form in ζ and η whose discriminant

$16\omega(\omega + \xi)$ is strictly positive by (3.6). The form is therefore positive definite and so the only solution to (3.10) is $\zeta = \eta = 0$. Hence we have $r_0 = r$ and $s_0 = s$. Furthermore, from the inequality

$$\left| s + r \left\{ \frac{\omega + i(t_0)}{\omega - i(t_0)} \right\} - f(\omega) \right|^2 \leqslant \delta^2$$

and (3.2) we obtain

$$\delta^2(1 - \omega)(\omega - (t_0)^2)/(\omega^2 + (t_0)^2) \leqslant 0,$$

and so $\omega \leqslant (t_0)^2$. On the other hand from

$$\left| s + r \left\{ \frac{1 + i(t_0)}{1 - i(t_0)} \right\} - 1 \right|^2 \leqslant \delta^2$$

we have

$$\delta^2(1 - \omega)((t_0)^2 - \omega)/\omega(1 + (t_0)^2) \leqslant 0,$$

and thus $(t_0)^2 \leqslant \omega$. We conclude that $(t_0)^2 = \omega = t^2$. This completes the proof.

Theorem B is now a simple corollary of Theorem A (and (2.2)).

4. Exactness of the Lower Bound

The result of the two preceding sections require nothing more of the function f than that it satisfy the standing hypotheses imposed in the first paragraph of Section 1. In the present section, however, continuity or differentiability will often be needed. Once continuity of f is required, the next result shows that Condition 2 of Section 1 is necessary for (1.8) to hold.

Theorem 4.1. *Let f be continuous on $[-1, 1]$ and suppose that there exists a symmetric transformation U such that $\| U - f \|_\infty = \Delta(f)$. Then*

(i) *there exists a unique point $\Omega \in (0, 1)$ such that $\Delta(f) = \delta(f, \Omega)$,*

(ii) *either $U(x) = U(x; f, \Omega)$ or $\overline{U(x)} = U(x; f, \Omega)$.*

Proof. The continuity of f implies that $\delta(f, \omega)$ is a continuous function of ω on $[0, 1]$. Hence $\delta(f, \omega)$ attains tis supremum $\Delta(f)$ at some point $\omega = \Omega$ in $[0, 1]$. But $\delta(f, \omega) = 0$ when $\omega = 0$ or $\omega = 1$ and yet $\delta(f, \omega)$ clearly assumes some positive values on $(0, 1)$. Hence we have $\Omega \in (0, 1)$.

From Theorem A we have

$$\begin{aligned}\Delta(f) &= \| U - f\|_\infty \\ &\geqslant \max_{x\in Z(\Omega)} | U(x) - f(x)| \\ &\geqslant \delta(f, \Omega) = \Delta(f).\end{aligned}$$

If follows that

$$\max_{x\in Z(\Omega)} | U(x) - f(x)| = \delta(f, \Omega),$$

and so by the uniqueness assertion in Theorem A we must have $U(x) = U(x; f, \Omega)$ or $\overline{U(x)} = U(x; f, \Omega)$.

If there is a second point $\Omega' \in (0, 1)$ for which $\Delta(f) = \delta(f, \Omega')$, then the same argument shows that $U(x) = U(x; f, \Omega')$ or $\overline{U(x)} = U(x; f, \Omega')$. Thus, modulo complex conjugation, the transformations $U(x; f, \Omega)$ and $U(x; f, \Omega')$ coincide. By Theorem 2.4 the parameters r, s and t^2 determined by these transformations must also coincide. In particular,

$$\Omega = t(f, \Omega)^2 = t(f, \Omega')^2 = \Omega'.$$

This shows that Ω is unique and completes the proof.

We now show that Condition 2 holds whenever f is continuous and convex. However, convexity is not necessary; in Section 5 we prove that $f_\alpha(x) = | x |^\alpha$ satisfies Condition 2 for all $\alpha > 0$.

THEOREM 4.2. *Suppose f is continuous and convex. Then there exists a unique point $\Omega \in (0, 1)$ such that $\Delta(f) = \delta(f, \Omega)$.*

Proof. As in the proof of Theorem 4.1 we know that $\delta(f, \omega)$ attains its supremum $\Delta(f)$ at some point $\Omega \in (0, 1)$. Suppose that $\Omega' \in (0, 1)$ is distinct from Ω and yet also satisfies $\Delta(f) = \delta(f, \Omega')$. Let $\Omega'' = \frac{1}{2}(\Omega + \Omega')$ and set

$$\phi(\omega) = 1 - \left(\frac{1 + \omega}{\omega^{1/2}}\right) \Delta(f), \qquad 0 < \omega < 1.$$

Then $\phi(\Omega) = f(\Omega)$ and $\phi(\Omega') = f(\Omega')$. Thus since ϕ is strictly concave and f is convex we have

$$\begin{aligned}\phi(\Omega'') &> \tfrac{1}{2}\{\phi(\Omega) + \phi(\Omega')\} \\ &= \tfrac{1}{2}\{f(\Omega) + f(\Omega')\} \geqslant f(\Omega'').\end{aligned}$$

But this implies $\Delta(f) < \delta(f, \Omega'')$, which is impossible. Hence Ω is unique.

LEMMA 4.3. *Let f satisfy Conditions* 1 *and* 2 *and let $M(x)$ be defined by*

$$M(x) = | U(x, f, \Omega) - f(x)|^2.$$

Then $M'(\Omega) = 0$.

Proof. The differentiability of f implies that the functions $\delta(f, x)$ and $\delta(f, x)^2$ are both differentiable on $(0, 1)$. Moreover, the derivative of $\delta(f, x)^2$ vanishes at $x = \Omega$ since $\delta(f, x)$, and hence $\delta(f, x)^2$, attains its maximum at $x = \Omega$.

By Theorem A we have $\delta(f, x)^2 \leqslant M(x)$ for all $x \in (0, 1)$, with equality if $x = \Omega$. Hence for $x > \Omega$ we obtain

$$\frac{M(x) - M(\Omega)}{x - \Omega} \geqslant \frac{\delta(f, x)^2 - \delta(f, \Omega)^2}{x - \Omega}. \tag{4.1}$$

Since $M'(x)$ exists on $(0, 1)$ we can let $x \to \Omega+$ in (4.1) and deduce that $M'(\Omega) \geqslant 0$. Similarly, by considering $x < \Omega$ we find that $M'(\Omega) \leqslant 0$. This completes the proof.

We now consider the proof of Theorem C. For any function f and any symmetric transformation U, the inequality (1.10) has already been established in Theorem B. If f is continuous and equality holds in (1.10). then by Theorem 4.1 we have either $U(x) = U(x; f, \Omega)$ or $\overline{U(x)} = U(x; f, \Omega)$. Thus to complete the proof we must show that under the hypotheses of Theorem C,

$$M(x) = | U(x; f, \Omega) - f(x)|^2 \leqslant \Delta(f)^2 \tag{4.2}$$

for $0 \leqslant x \leqslant 1$. We note that Theorem 2.3 then implies that $\frac{1}{4} < \Delta(f)$. In fact, we shall prove slightly more than (4.2) in the following theorem.

THEOREM 4.4. *Let f be a function which satisfies the hypotheses of Theorem C. Then*

(i) *the inequality* (4.2) *holds,*

(ii) *f satisfies the inequality*

$$0 \leqslant 1 - 2f(\Omega^{1/2}) + f(\Omega), \tag{4.3}$$

(iii) *if there is strict inequality in* (4.3), *then equality occurs in* (4.2) *if and only if $x = \Omega$ or $x = 1$.*

(iv) *if equality occurs in* (4.3) *then $\Omega = 3 - 2\sqrt{2}$ and there exists a point $\Omega_0 \in [0, \Omega]$ such that equality occurs in* (4.2) *if and only if $\Omega_0 \leqslant x \leqslant 1$,*

(v) *equality occurs in* (4.3) *and* $\Omega_0 = 0$ *if and only if*

$$f(x) = F(x) = \frac{2(2-\sqrt{2})\,x^2}{x^2+3-2\sqrt{2}}. \tag{4.4}$$

Proof. Since $\delta(f, \omega) = \omega^{1/2}(1 - f(\omega))/(1 + \omega)$ takes some positive values we must have $0 \leqslant f(\Omega) < 1$. Thus

$$U(x; f, \Omega) = s + r\left\{\frac{x + it}{x - it}\right\},$$

where r, s and t are given by (3.2) with $\omega = \Omega$. We then have

$$M(x) = \frac{x^2}{x^2+\Omega}\{f(x) - r - s\}^2 + \frac{\Omega}{x^2+\Omega}\{f(x) + r - s\}^2.$$

It will be convenient to introduce the change of variables

$$\lambda = \lambda(x) = x^2/(x^2 + \Omega), \tag{4.5}$$

$$x = x(\lambda) = (\Omega\lambda/(1-\lambda))^{1/2}. \tag{4.6}$$

We then set

$$g(\lambda) = f(x(\lambda)) \tag{4.7}$$

and

$$\begin{aligned} N(\lambda) &= M(x(\lambda)) \\ &= \lambda\{g(\lambda) - r - s\}^2 + (1-\lambda)\{g(\lambda) + r - s\}^2. \end{aligned} \tag{4.8}$$

Also, we define $\xi = \lambda(\Omega) = \Omega/(1 + \Omega)$ so that $0 < \xi < 1/2$ and the range $0 \leqslant x \leqslant 1$ corresponds to $0 \leqslant \lambda \leqslant 1 - \xi$.

From Theorem A we have

$$N(\xi) = M(\Omega) = \Delta(f)^2 = M(1) = N(1 - \xi), \tag{4.9}$$

and from Lemma 4.3,

$$N'(\xi) = 0. \tag{4.10}$$

There is also a point $\rho \in (\xi, 1 - \xi)$ such that

$$N'(\rho) = 0. \tag{4.11}$$

This is easily seen from (4.9) and Rolle's theorem. The inequality (4.2) which we wish to prove is now equivalent to

$$N(\lambda) \leqslant \Delta(f)^2, \qquad 0 \leqslant \lambda \leqslant 1 - \xi. \tag{4.12}$$

In order to establish (4.12) we shall make a careful examination of the sign of $N'(\lambda)$ and show that $N(\lambda)$ always assumes its global maximum at $\lambda = \xi$ and $1 - \xi$.

Differentiating (4.7) with respect to λ we have for $0 < \lambda < 1 - \xi$,

$$g'(\lambda) = \frac{(x^2 + \Omega)^2}{2\Omega x} f'(x).$$

Thus the hypothesis (1.9) implies that $g'(\lambda)$ is non-negative, increasing and hence continuous. We also deduce that $g(\lambda)$ is convex on $[0, 1 - \xi]$. Since $g(0) = 0$, this shows that $g(\lambda)$ can vanish only for those λ in a closed interval $[0, \beta]$ with $0 \leqslant \beta < 1 - \xi$. We also note that $g'(\lambda) = 0$ if and only if $g(\lambda) = 0$.

Next we consider the functions

$$\psi_1(\lambda) = g(\lambda) - s, \qquad \psi_2(\lambda) = g(\lambda) - s + (1 - 2\lambda)r. \tag{4.13}$$

The derivatives $\psi_1' = g'$ and $\psi_2' = g' - 2r$ are increasing so both ψ_1 and ψ_2 are continuous convex functions on $[0, 1 - \xi]$. From (4.13) we have

$$\psi_1(0) = -s < 0, \qquad \psi_1(1 - \xi) = 1 - s > 0$$

and

$$\psi_2(0) = r - s < 0, \qquad \psi_2(1 - \xi) = 1 - r - s + 2\xi r > 0.$$

Hence there exist unique points λ_1 and λ_2 in $(0, 1 - \xi)$ such that

$$\begin{aligned} \psi_j(\lambda) &< 0 \quad \text{if} \quad 0 \leqslant \lambda < \lambda_j, \\ \psi_j(\lambda) &= 0 \quad \text{if} \quad \lambda = \lambda_j, \\ \psi_j(\lambda) &> 0 \quad \text{if} \quad \lambda_j < \lambda \leqslant 1 - \xi, \end{aligned} \tag{4.14}$$

for $j = 1, 2$.

Another consequence of the convexity of g is the inequality

$$\begin{aligned} f(\Omega^{1/2}) &= g(1/2) \leqslant \tfrac{1}{2}g(\xi) + \tfrac{1}{2}g(1 - \xi) \\ &= \tfrac{1}{2}(f(\Omega) + 1) = s, \end{aligned} \tag{4.15}$$

which is equivalent to (4.3). We now divide the proof into two cases according to whether equality or inequality occurs in (4.15).

Case I. Suppose $g(1/2) < s$. Then $\lambda_1 > 1/2$, and since

$$\begin{aligned}\psi_2(\lambda_1) &= g(\lambda_1) - s + (1 - 2\lambda_1)r \\ &= (1 - 2\lambda_1)r < 0,\end{aligned}$$

we must have $\frac{1}{2} < \lambda_1 < \lambda_2$. Differentiating (4.8) with respect to λ we obtain

$$N'(\lambda) = -4r\psi_1(\lambda) + 2g'(\lambda)\,\psi_2(\lambda). \tag{4.16}$$

Thus $N'(\lambda_2) = -4r\psi_1(\lambda_2) < 0$, and if $\lambda \neq \lambda_2$ then $N'(\lambda) = 0$ if and only if

$$g'(\lambda) = \frac{2r\psi_1(\lambda)}{\psi_2(\lambda)}. \tag{4.17}$$

Now suppose for the moment that we can show that $\psi_1(\lambda)/\psi_2(\lambda)$ is strictly decreasing on each of the intervals $(0, \lambda_2)$ and $(\lambda_2, 1 - \xi)$. Then on each of these intervals, the left-hand side of (4.17) is increasing and the right-hand side is strictly decreasing. It follows that $N'(\lambda)$ has at most one zero in $(0, \lambda_2)$, which must be ξ, and at most one zero in $(\lambda_2, 1 - \xi)$, which must be ρ. This analysis also shows that $g'(\lambda)$ is less than $2r\psi_1(\lambda)/\psi_2(\lambda)$ on the set $(0, \xi) \cup (\lambda_2, \rho)$, and greater than $2r\psi_1(\lambda)/\psi_2(\lambda)$ on the set $(\xi, \lambda_2) \cup (\rho, 1 - \xi)$. Hence from (4.16) we deduce that $N'(\lambda)$ is strictly positive on $(0, \xi)$, is zero at $\lambda = \xi$, is strictly negative on (ξ, ρ), is zero at $\lambda = \rho$, and is strictly positive on $(\rho, 1 - \xi)$. From this we conclude that $N(\lambda)$ can assume its maximum value only at $\lambda = \xi$ or $\lambda = 1 - \xi$. In view of (4.9), this establishes the desired inequality (4.12).

Thus it remains only to show that ψ_1/ψ_2 is strictly decreasing on $(0, \lambda_2)$ and on $(\lambda_2, 1 - \xi)$. From (4.13) we have

$$\frac{d}{d\lambda}\left(\frac{\psi_1(\lambda)}{\psi_2(\lambda)}\right) = r\,\frac{(1 - 2\lambda)\,g'(\lambda) + 2(g(\lambda) - s)}{(g(\lambda) - s + (1 - 2\lambda)r)^2}, \tag{4.18}$$

for $\lambda \neq \lambda_2$. If $g'(\lambda) = 0$ then $g(\lambda) = 0$ and so the right-hand side of (4.18) is negative, as required. If $g'(\lambda) > 0$, we argue as follows. By the mean-value theorem and the fact that $g(\lambda_1) = s$ we have

$$\begin{aligned}&(1 - 2\lambda)\,g'(\lambda) + 2(g(\lambda) - s) \\ &\quad = (1 - 2\lambda)\,g'(\lambda) + 2g'(\nu)(\lambda - \lambda_1),\end{aligned} \tag{4.19}$$

for some ν lying between λ and λ_1. But g' is increasing so that the right-hand side of (4.19) is less than or equal to

$$(1 - 2\lambda)\,g'(\lambda) + 2g'(\lambda)(\lambda - \lambda_1) = g'(\lambda)(1 - 2\lambda_1).$$

Since $\lambda_1 > 1/2$ we must have $g'(\lambda)(1 - 2\lambda_1) < 0$ and thus (4.18) is negative. This shows that ψ_1/ψ_2 is decreasing and completes the proof in Case I.

Case II. Suppose $g(1/2) = s$. Then $\lambda_1 = 1/2$ and since $\psi_2(1/2) = 0$ we also have $\lambda_2 = 1/2$. Let $A(\lambda)$ be the linear function

$$A(\lambda) = (1 - 2\lambda)\frac{g(\xi) - s}{1 - 2\xi} + s.$$

Then $A(\xi) = g(\xi)$ and $A(1 - \xi) = g(1 - \xi) = 1$. Since g is continuous, convex and

$$g(1/2) = s = \tfrac{1}{2}g(\xi) + \tfrac{1}{2}g(1 - \xi)$$

we must have $g(\lambda) = A(\lambda)$ at least for $\xi \leqslant \lambda \leqslant 1 - \xi$. In fact there exists a smallest real number $\lambda_0 \in [0, \xi]$ such that $g(\lambda) = A(\lambda)$ on $[\lambda_0, 1 - \xi]$ and (if $\lambda_0 \neq 0$) $g(\lambda) > A(\lambda)$ on $[0, \lambda_0)$. We see from (4.16) that the derivative $N'(\lambda)$ is linear on $[\lambda_0, 1 - \xi)$. But $N'(\xi) = 0 = N'(1/2)$ so that $N'(\lambda)$ is identically zero on $[\lambda_0, 1 - \xi)$. Thus $N(\lambda)$ is constant on $[\lambda_0, 1 - \xi]$ and hence by (4.9),

$$N(\lambda) = \Delta(f)^2, \qquad \lambda_0 \leqslant \lambda \leqslant 1 - \xi. \tag{4.20}$$

If we define $\Omega_0 = x(\lambda_0)$ then we have the corresponding result $M(x) = \Delta(f)^2$ for $\Omega_0 \leqslant x \leqslant 1$.

If $\lambda_0 = 0$ then (4.20) establishes (4.12). If $\lambda_0 \neq 0$ then for $0 \leqslant \lambda < \lambda_0$ we have $g(\lambda) > A(\lambda)$, or equivalently,

$$\frac{g(\lambda) - s}{1 - 2\lambda} > \frac{g(\xi) - s}{1 - 2\xi}.$$

Combining this with (4.13) we obtain

$$\frac{2r\psi_1(\lambda)}{\psi_2(\lambda)} > \frac{2r\psi_1(\xi)}{\psi_2(\xi)}.$$

Using $N'(\xi) = 0$, (4.17) and the fact that g' increases, we deduce that

$$g'(\lambda) \leqslant g'(\xi) = \frac{2r\psi_1(\xi)}{\psi_2(\xi)} < \frac{2r\psi_1(\lambda)}{\psi_2(\lambda)}$$

for $0 < \lambda < \lambda_0$. Thus by (4.16) we have $N'(\lambda) > 0$ when $0 < \lambda < \lambda_0$. This shows that $N(\lambda)$ increases on $[0, \lambda_0)$ and is constant and equal to $\Delta(f)^2$ on $[\lambda_0, 1 - \xi]$. Hence (4.12) is established.

At $\lambda = 1/2$ we see from (4.20) that $\Delta(f)^2 = N(1/2) = r^2$. Now (1.3) and (3.2) lead to the identity

$$\frac{(1-\Omega)(1-f(\Omega))}{2(1+\Omega)} = \frac{\Omega^{1/2}(1-f(\Omega))}{1+\Omega},$$

which has the unique solution $\Omega = 3 - 2\sqrt{2}$. We have thus established the first four parts of the theorem.

For part (v) we carry the analysis of Case II above one step further. We showed that $g(\lambda)$ coincides with $A(\lambda)$ when $\lambda_0 \leqslant \lambda \leqslant 1 - \xi$. In terms of the function f, this asserts that

$$f(x) = \frac{\Omega(1-x^2)f(\Omega) + x^2 - \Omega^2}{(1-\Omega)(x^2+\Omega)}, \qquad \Omega_0 \leqslant x \leqslant 1. \qquad 4.21)$$

If $\Omega_0 = 0 = \lambda_0$ then $g(\lambda)$ and $A(\lambda)$ coincide for all $\lambda \in [0, 1-\xi]$. In particular

$$0 = g(0) = A(0) = \frac{g(\xi) - 2\xi s}{1 - 2\xi} = \frac{f(\Omega) - \Omega}{1 - \Omega}.$$

Thus $f(\Omega) = \Omega = 3 - 2\sqrt{2}$ and (4.21) reduces to (4.4). This establishes the "only if" assertion in part (v).

Finally we must show that if $f = F$ is defined by (4.4) then the hypotheses of the theorem are satisfied, $\Omega = 3 - 2\sqrt{2}$ and $\Omega_0 = 0$. There is no difficulty in verifying Condition 1. For Condition 2 we have

$$\delta(F, \omega) = \frac{(3 - 2\sqrt{2})\,\omega^{1/2}(1-\omega)}{\omega^2 + 3 - 2\sqrt{2}}, \qquad 0 \leqslant \omega \leqslant 1,$$

and

$$\delta'(F, \omega) = \frac{(3-2\sqrt{2})\{\omega^3 - 3\omega^2 - (3-2\sqrt{2})(3\omega - 1)\}}{2\omega^{1/2}(\omega^2 + 3 - 2\sqrt{2})^2},$$

$$0 < \omega < 1. \qquad (4.22)$$

The cubic polynomial in the numerator of (4.22) has roots at $\omega = 3 - 2\sqrt{2}$ and $\omega = \sqrt{2} \pm \sqrt{3}$. Only the first of these is in (0, 1) and hence must be the unique point Ω at which $\delta(F, \omega)$ attains its supremum. We also note that (1.9) is easily seen to hold for the function F.

If $f = F$ then the corresponding function $g(\lambda)$ is simply $g(\lambda) = (4 - 2\sqrt{2})\lambda$. Since this holds for $0 \leqslant \lambda \leqslant 1 - \xi$ we must have $\lambda_0 = \Omega_0 = 0$. This completes the proof of part (v).

5. The Functions $|x|^\alpha$

In this section we shall restrict our attention to approximation of the functions $f_\alpha(x) = |x|^\alpha$, $\alpha > 0$. These functions clearly satisfy Condition 1 and we now show that they satisfy Condition 2.

THEOREM 5.1. *Let $\alpha > 0$ and let*

$$D_\alpha(\omega) = 1 - \omega - (2\alpha + 1)\,\omega^\alpha - (2\alpha - 1)\,\omega^{\alpha+1} \tag{5.1}$$

for $0 \leqslant \omega \leqslant 1$. Then there exists a unique point $\Omega_\alpha \in (0, 1)$ such that $\Delta(f_\alpha) = \delta(f_\alpha, \Omega_\alpha)$. Moreover, $\omega = \Omega_\alpha$ is the unique root in $[0, 1]$ of $D_\alpha(\omega) = 0$.

Proof. Since $\delta(f_\alpha, \omega)$ vanishes at $\omega = 0$ and $\omega = 1$, it will suffice to show that $(d/d\omega)\,\delta(f_\alpha, \omega)$ has a unique zero in $(0, 1)$. This is easily seen to be equivalent to showing that $D_\alpha(\omega)$ has a unique zero in $(0, 1)$. We have $D_\alpha(0) = 1 > 0$, $D_\alpha(1) = -4\alpha < 0$ and

$$\begin{aligned} \frac{d}{d\omega} D_\alpha(\omega) &= -1 - \omega^\alpha\{\alpha(2\alpha + 1)\,\omega^{-1} + (\alpha + 1)(2\alpha - 1)\} \\ &< -1 + \omega^\alpha < 0 \end{aligned} \tag{5.2}$$

for $0 < \omega < 1$. Thus $D_\alpha(\omega)$ has a unique zero in $[0, 1]$.

THEOREM 5.2. *For $0 < \alpha < \infty$, Ω_α is a continuous, strictly increasing function of α. Also we have*

(i) $\lim_{\alpha\to\infty} \Omega_\alpha = 1$,

(ii) $\lim_{\alpha\to 0+} \Omega_\alpha = \tau$,

where $\tau = .09077628 \cdots$ is the unique root in $(0, 1)$ of

$$\frac{2(1 + \tau)}{1 - \tau} + \log \tau = 0.$$

Proof. The function $D_\alpha(\omega)$ defined in (5.1) is clearly a continuous function of the two variables α and ω for $(\alpha, \omega) \in (0, \infty) \times [0, 1]$. Thus the subset $\{(\alpha, \omega): D_\alpha(\omega) = 0\} = \{(\alpha, \Omega_\alpha): 0 < \alpha < \infty\}$ is closed and so Ω_α is a continuous function of α.

For fixed $\alpha \in (0, \infty)$, the estimate (5.2) shows that $D_\alpha(\omega)$ decreases on $[0, 1]$. Hence the inequality $D_\alpha(\omega) > 0$ is equivalent to $\omega < \Omega_\alpha$.

Next let $0 < \alpha < \beta < \infty$ and let x be a real variable. Then the curve $(\Omega_\alpha)^{-x}$ intersects the straight line

$$y = \frac{2(1 + \Omega_\alpha)x}{1 - \Omega_\alpha} + 1$$

in exactly the two points $x = 0$ and $x = \alpha$. The case $x = 0$ is trivial and the case $x = \alpha$ follows immediately from $D_\alpha(\Omega_\alpha) = 0$. Since $(\Omega_\alpha)^{-x}$ is convex we have

$$(\Omega_\alpha)^{-x} > \frac{2(1+\Omega_\alpha)}{1-\Omega_\alpha} x + 1, \qquad \alpha < x < \infty,$$

or equivalently $D_x(\Omega_\alpha) > 0$, $\alpha < x < \infty$. In particular, if $x = \beta$ then $D_\beta(\Omega_\alpha) > 0$ or $\Omega_\alpha < \Omega_\beta$, as required.

The line

$$y = \frac{2(1+\tau)x}{1-\tau} + 1$$

is clearly tangent to the curve τ^{-x} at $x = 0$. Since τ^{-x} is convex,

$$\tau^{-x} > \frac{2(1+\tau)x}{1-\tau} + 1, \qquad 0 < x < \infty,$$

and so $D_x(\tau) > 0$ for $0 < x < \infty$. Thus if $\tau^* = \lim_{\alpha\to 0+} \Omega_\alpha$ (which must exist since Ω_α is monotone) then

$$0 < \tau \leqslant \tau^*. \tag{5.3}$$

On the other hand the mean-value theorem shows that there exists a point $\nu \in (0, \alpha)$ such that

$$(\Omega_\alpha)^{-\nu}(-\log \Omega_\alpha) = \frac{2(1+\Omega_\alpha)}{1-\Omega_\alpha}. \tag{5.4}$$

Taking limits on both sides of (5.4) and using (5.3), we obtain

$$-\log \tau^* = \frac{2(1+\tau^*)}{1-\tau^*}.$$

Thus $\tau^* = \tau$ and (ii) is proved.

To prove (i) we note that $\lim_{\alpha\to\infty} \Omega_\alpha = \sigma^*$ exists by monotonicity and satisfies $\sigma^* \leqslant 1$. If $\sigma^* < 1$ then taking limits as $\alpha \to \infty$ on both sides of

$$1 - \Omega_\alpha - (2\alpha+1)(\Omega_\alpha)^\alpha - (2\alpha-1)(\Omega_\alpha)^{\alpha+1} = 0$$

produces $1 - \sigma^* = 0$, which is impossible. Thus $\sigma^* = 1$ and (i) is proved.

THEOREM 5.3. *Let κ be the unique real number in* $(1, \infty)$ *which satisfies* (1.11).

(i) *If* $\alpha = \kappa$ *then* $|U(0; f_\alpha, \Omega_\alpha)| = \Delta(f_\alpha)$,

(ii) *if* $\alpha > \kappa$ *then* $|U(0; f_\alpha, \Omega_\alpha)| < \Delta(f_\alpha)$,

(iii) *if* $\alpha < \kappa$ *then* $|U(0; f_\alpha, \Omega_\alpha)| > \Delta(f_\alpha)$.

Proof. First of all we observe that by Theorem 5.2,

$$|U(0; f_\alpha, \Omega_\alpha)| - \Delta(f_\alpha) \tag{5.5}$$

is a continuous function of α for $0 < \alpha < \infty$. A simple computation shows that (5.5) is positive at $\alpha = 1$ and negative at $\alpha = 2$. Thus it suffices to prove that (5.5) has a unique zero at $\alpha = \kappa = 1.4397589 \cdots$.

Throughout the remainder of this proof it will be convenient to write Ω instead of Ω_α. If we set (5.5) equal to zero and use (3.1), (3.2) and (1.3) we obtain

$$(1 + \Omega)(1 + \Omega^\alpha) - (1 - \Omega)(1 - \Omega^\alpha) = 2\Omega^{1/2}(1 - \Omega^\alpha). \tag{5.6}$$

By Theorem 5.1 we have $D_\alpha(\Omega) = 0$ or equivalently

$$\Omega^\alpha = \frac{(1 - \Omega)}{2(1 + \Omega)\alpha + (1 - \Omega)}. \tag{5.7}$$

Substituting (5.7) into (5.6) we obtain

$$2(1 + \Omega)^2 \alpha + 2(1 - \Omega^2) - 2(1 - \Omega^2)\alpha = 4\alpha\Omega^{1/2}(1 + \Omega). \tag{5.8}$$

If we divide both sides of (5.8) by $(1 + \Omega)$ and write $(1 - \Omega)$ as $(1 - \Omega^{1/2})(1 + \Omega^{1/2})$ then (5.8) reduces to

$$\Omega^{-1/2} = 2\alpha - 1. \tag{5.9}$$

Since $0 < \Omega < 1$, if follows from (5.9) that $\alpha > 1$. Next we substitute $\Omega = (2\alpha - 1)^{-2}$ into $D_\alpha(\Omega) = 0$ and multiply both sides by $(2\alpha - 1)^{2\alpha+1}$ to produce

$$(2\alpha - 1)^{2\alpha+1} - (2\alpha - 1)^{2\alpha-1} - (2\alpha + 1)(2\alpha - 1) - 1 = 0$$

or equivalently

$$(2\alpha - 1)^{2\alpha-1} = \frac{\alpha}{\alpha - 1}. \tag{5.10}$$

The function $(2\alpha - 1)^{2\alpha-1}$ increases on $(1, \infty)$ from 1 to ∞ while $\alpha/(\alpha - 1)$ decreases from ∞ to 1. Thus (5.10) has a unique solution κ. Reversing the previous calculations we see that $\alpha = \kappa$ is the unique value of α for which (5.5) is zero.

We are now ready to begin our proof of Theorem D. First we consider the case $0 < \alpha < \kappa$. From Theorem B we have

$$\| U - f_\alpha \|_\infty \geqslant \Delta(f_\alpha) \tag{5.11}$$

for any symmetric transformation U. By Theorem 4.1 equality can occur in (5.11) only if $U(x) = U(x; f_\alpha, \Omega_\alpha)$ or $\overline{U(x)} = U(x; f_\alpha, \Omega_\alpha)$. But now Theorem 5.3 shows that if $0 < \alpha < \kappa$ then

$$\sup_{-1 \leqslant x \leqslant 1} | U(x; f_\alpha, \Omega_\alpha) - f_\alpha(x)| \geqslant | U(0; f_\alpha, \Omega_\alpha)| > \Delta(f_\alpha),$$

with an identical inequality for the conjugate transformation. This together with Theorem 2.3 completes the proof of Theorem D part (ii).

Next we observe that if $2 \leqslant \alpha < \infty$ then (1.9) is increasing. Thus for α in this range Theorem D follows directly from Theorem C. We also note that there is strict inequality in (4.3).

If $\kappa \leqslant \alpha < 2$ then f_α fails to satisfy the additional hypothesis of Theorem C. However, the techniques used in the proof of Theorem 4.4 remain applicable. The key difference is that the function $g(\lambda) = f_\alpha(x(\lambda))$ defined by (4.7) no longer has an increasing derivative on $(0, 1 - \xi)$. Rather, $g'(\lambda)$ increases on $((2 - \alpha)/4, 1 - \xi)$ but decreases on $(0, (2 - \alpha)/4)$. As before we can show that on $((2 - \alpha)/4, 1 - \xi)$ the function $N(\lambda)$ (cf. (4.8)) has a relative maximum at $\xi \geqslant (2 - \alpha)/4$ and a relative minimum on $(\xi, 1 - \xi)$. On the interval $(0, (2 - \alpha)/4)$ a careful analysis shows that $N(\lambda)$ has one extremum, a relative minimum. We exclude the details. Theorem 5.3 then shows that (4.12) holds and the proof of Theorem D is complete.

We remark that if $\kappa < \alpha$ then equality holds in

$$| U(x; f_\alpha, \Omega_\alpha) - f_\alpha(x)| \leqslant \Delta(f_\alpha) \tag{5.23}$$

if and only if $x \in Z(\Omega_\alpha)$. However if $\kappa = \alpha$ then there is also equality in (5.23) at $x = 0$. We suspect that this may provide a clue to the behavior of the extremal transformations for $\alpha < \kappa$. Namely, that if $\alpha < \kappa$ and U_α is a symmetric transformation such that $\| U_\alpha - f_\alpha \|_\infty = E_S(f_\alpha)$, then there is equality in the inequality

$$| U_\alpha(x) - f_\alpha(x)| \leqslant E_S(f_\alpha), \qquad -1 \leqslant x \leqslant 1,$$

if and only if $x \in \{-1, -\zeta, 0, \zeta, 1\}$ for some $\zeta \in (0, 1)$ which depends on $\boldsymbol{\alpha}$. It seems likely that such a result holds at least for α sufficiently close to κ. At present, however, all that we can prove for $\alpha < \kappa$ is the inequality (1.12).

We conclude our discussion of f_α with a number-theoretic result which follows simply from (1.11) and the Gelfond-Schneider Theorem (cf. [5, pp. 80–83]).

THEOREM 5.4. *The constant κ is transcendental.*

TABLE 1

α	Ω_α	$\Delta(f_\alpha)$
1	0.236068	0.300283
κ	0.283079	0.347281
2	0.333333	0.384900
3	0.404214	0.422862
4	0.458819	0.443745
5	0.502528	0.456679
10	0.636836	0.482189
100	0.918863	0.499447

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